

Factorized soft graviton theorems at loop level

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Abstract

We analyze the low-energy behavior of scattering amplitudes involving gravitons at loop level in four dimensions. The single-graviton soft limit is controlled by soft operators which have been argued to separate into a factorized piece and a non-factorizing infrared divergent contribution. In this note we show that the soft operators responsible for the factorized contributions are strongly constrained by gauge and Poincaré invariance under the assumption of a local structure. We show that the leading and subleading orders in the soft-momentum expansion can not receive radiative corrections. The first radiative correction occurs for the sub-subleading soft graviton operator and is one-loop exact. It depends on only two undetermined coefficients which should reflect the field content of the theory under consideration.

1 Introduction

Recently the low energy behavior of graviton scattering amplitudes with a single soft graviton momentum has been related [1] to Ward identities of the extended Bondi, van der Burg, Metzner and Sachs (BMS) symmetry [2]. The role of BMS symmetry as a potential hidden symmetry of the quantum gravity S-matrix in asymptotically flat four-dimensional space-time has triggered considerable interest.

In the soft limit the $(n + 1)$ -point scattering amplitude is dominated by a soft pole as was shown by Weinberg about fifty years ago [3]. The soft amplitude factorizes on the pole into a universal soft function and the remaining hard graviton n -point amplitude. This property holds in any spacetime dimension.

Revived by the work of Cachazo and Strominger [4] the universal factorization has been shown to extend to sub- and sub-subleading order in the soft-momentum expansion [5]. In distinction to the leading Weinberg pole, which is a function of the soft and hard momenta and the soft graviton polarization, the sub- and sub-subleading soft behavior of the $(n + 1)$ -point graviton amplitude can be expressed in terms of differential operators in the hard momenta and hard graviton polarizations acting on the hard n -point amplitude.

Gluon amplitudes exhibit a similar universal leading and subleading soft behavior at tree level. Known as Low's theorem [6], the form of the soft operators was recently recast into the language of modern on-shell techniques [7]. The soft limit of open-string tree amplitudes was studied in refs. [8, 9], where further corrections to the field-theory soft theorems have been excluded. Interestingly, the subleading soft theorems for gravitons and gluons may also be derived from the soft limit of vertex operators in a dual ambitwistor string theory [10].

In a series of recent papers [11, 12] the existence of subleading soft-graviton and soft-gluon theorems was shown to be a consequence of on-shell gauge invariance and Feynman diagrammatic reasoning. Simultaneously, an alternative point of view was put forward by the present authors in ref. [13]: The tree-level soft-graviton and soft-gluon expressions are composed from a highly restricted class of operators compatible with on-shell gauge invariance, factorization and Poincaré symmetry. This result could be established without referring to underlying Feynman diagrammatics at all. For the graviton case, this reasoning supplements the established sub- and sub-subleading soft operators with one further candidate each.

Given these results it is natural to ask whether the soft-graviton theorems receive corrections at loop level [14, 15]. In four-dimensional gravity the dimension of the coupling constant κ leads to strong constraints on the possible loop corrections. While the leading soft function is known to be free of radiative corrections [16], the subleading and sub-subleading operators do not receive corrections beyond one and two loops respectively, as argued in [14, 15]. Therefore – upon controlling the soft limit of leg n with momentum ϵq^μ by the parameter ϵ – the L -loop graviton amplitude should behave as

$$\mathcal{M}_n^{L\text{-loop}} \xrightarrow{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon} S_G^{(0)} + S_G^{(1)} + \epsilon S_G^{(2)} \right) \mathcal{M}_{n-1}^{L\text{-loop}} \quad (1)$$

$$+ (S_G^{(1)1\text{-loop}} + \epsilon S_G^{(2)1\text{-loop}}) \mathcal{M}_{n-1}^{(L-1)\text{-loop}} + \epsilon S_G^{(2)2\text{-loop}} \mathcal{M}_{n-1}^{(L-2)\text{-loop}}, \quad (2)$$

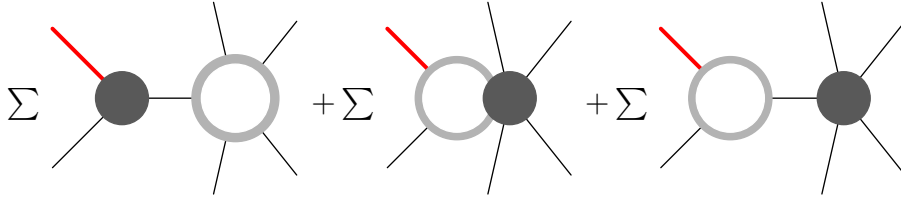


Figure 1: One-loop contributions to the soft expansion: the leading soft factor acting on the one-loop amplitude, the non-factorizing contributions and the factorizing contributions.

where \mathcal{M}_n denotes the graviton amplitude and $S_G^{(i)}$ the leading ($i = 0$), subleading ($i = 1$) or sub-subleading ($i = 2$) soft graviton operators at various loop levels¹.

In ref. [17] it was proposed that loop corrections to soft theorems could be generally suppressed by taking the soft limit prior to removing the dimensional regulator. Effectively this amounts to a study of the soft limit at the level of the loop integrand. However, such an order of limits appears physically unjustified to us. Moreover, as argued in refs. [14, 11] and extensively shown in ref. [18], this order of limits would invalidate the cancellation of leading IR divergences when applied to QCD.

A central point in the analysis of loop corrections to soft operators concerns the expected separation into two distinct pieces [11]: The “factorizing” and the “non-factorizing” contributions. The factorizing pieces correspond to the first and the last graph in figure 1: while the leading soft factor acting on the one-loop amplitude is depicted in the first graph, the last graph shows loop-corrected soft operators acting on the hard amplitude. The remaining middle graph is associated to the non-factorizing contributions.

The separation into factorizing and non-factorizing contributions was initially laid out for collinear limits in [19]. Working in dimensional regularization the poles in ϵ are to be subtracted from the factorized diagrams and attributed to the non-factorized contributions to the amplitude. Related to infrared divergences, the latter are under good control. This structure was established in gauge theory in [19, 20]. To our understanding a proof of an analogous separation into factorizing and non-factorizing pieces does not exist for gravitational amplitudes yet. Given the results for gauge theory, however, it appears highly plausible to exist at least at the one-loop level. An analysis of the graviton soft operators responsible for the leading divergent contributions to the operators $S_G^{(i)}$ was performed up to the two-loop level in [14].

Two different concepts are important for the description of soft operators in the current article: “locality” and “universality”. A soft operator is said to be local, if the soft particle interacts with each hard particle separately. Universality of a soft operator implies that the operator is of the same form for any amplitude, independent of the helicity configuration in question. In the current article, this universality is implemented by expressing all soft operators in terms of differential operators in momenta and polarization vectors. Accordingly, spinor-helicity expressions appearing in a calculation of the soft behavior might differ for different helicity

¹In the one-loop case the last term of course drops out, similarly at tree-level the last two terms are absent.

configurations despite originating from a single universal operator expressed in polarization and momentum vectors.

Related to the discussion of locality and universality above, an important subtlety was discussed in ref. [11]²: Contrary to the naive expectation, the factorizing contributions in gauge theory and gravity do contain *non-local* parts. While in the gauge-theory scenario this means that the soft particle interacts with non-neighboring legs, it implies interaction with more than one hard leg at a time for gravitational theories. While those contributions can be determined for every particular configuration, a general formula reproducing those parts is not known. Accordingly, the analysis in the current article will be limited to the universal, local and factorizing contributions.

In the present note we extend our tree-level analysis [13] of the operators appearing in soft-graviton theorems to loop level. In order to do so, we will *assume* that the separation into a factorizing and a non-factorizing part extends to higher loops for gravitons in four dimensions. Starting again from on-shell gauge invariance, factorization and Poincaré symmetry, we adapt the formalism to the appropriate mass dimensions at different loop levels and identify the restricted class of operators which can appear at leading, subleading and sub-subleading order. A crucial additional assumption is again the “local” form of the soft operators explained above. As our method allows to constrain operators composed of hard polarizations and momenta only, it is blind to possible non-universal contributions to the factorized part.

Before moving on to gravity, let us comment on the situation in four-dimensional gauge theory. Here there are no novel loop-level operators arising from our analysis because of the dimensionless coupling constant. This implies in particular that the new subleading operator identified in [13] captures the form of all radiative corrections. In fact, the form reported in eq. (34) of [13] coincides with the one-loop contribution quoted in [11]. The undetermined coefficient depends on the matter content of the gauge theory in question, whereas the tensorial form of the subleading soft-gluon operator is universal. However, it is precisely at this order where one first encounters the *non-local* parts mentioned above: As was analyzed carefully in section 3.2 of [15] the single-minus one-loop $(n + 1)$ -gluon amplitude with a soft leg of positive helicity develops a factorized *non-local* subleading soft pole, whose corresponding operator depends on next-to-nearest neighboring hard legs of the soft leg. Unfortunately our formalism is unable to capture this behavior at present.

2 Method and previous results

For completeness we repeat our method and the resulting soft-graviton operators at tree-level [13]. We denote four-dimensional graviton amplitudes by $\mathcal{M} = \delta^{(4)}(P) M$ where P is the total momentum. The momentum of leg $(n + 1)$ is expressed as ϵq , which can be made soft by sending $\epsilon \rightarrow 0$. The remaining hard momenta are labeled p_a with $a = 1, \dots, n$. Moreover, we write the polarization tensor of the soft leg as $E_{\mu\nu} = E_\mu E_\nu$ and similarly for the hard legs

²We would like to thank Zvi Bern, Paolo di Vecchia, Josh Nohle and Scott Davies for pointing out and explaining this subtlety.

$(E_a)_{\mu\nu} = E_{a,\mu} E_{a,\nu}$. This splitting is an identity in four dimensions. In this language, the soft-graviton theorem [4] reads

$$\mathcal{M}_{n+1}(p_1, \dots, p_n, \epsilon q) = \left(\frac{1}{\epsilon} S_G^{(0)} + S_G^{(1)} + \epsilon S_G^{(2)} \right) \mathcal{M}_n(p_1, \dots, p_n) + \mathcal{O}(\epsilon^2), \quad (3)$$

where we have suppressed the polarization dependence of the amplitudes \mathcal{M}_m . The soft operators $S_G^{(l)}$ are in general differential operators in the hard momenta p_a and polarizations E_a and also depend on the soft data q and E .

2.1 Constraints on factorized soft operators

We recall here the assumptions on the form of the soft operators and the constraints they must satisfy, both of which were analyzed in detail in ref. [13].

Locality. The only assumption on the form of the soft terms is what we termed *locality*. We assume that the soft operators $S_G^{(l)}$ can be expressed as a sum of local operators, each of which acts on the soft leg and one single hard leg:

$$S_G^{(l)} = \sum_{a=1}^n S_a^{(l)}(E, q, E_a, p_a; \partial_{p_a}, \partial_{E_a}). \quad (4)$$

Clearly this assumption is justified at tree level. At loop level, however, there are contributions which are either non-local or non-universal or even both, as shown in explicit computations of the IR divergences in refs. [11, 15].

Distributional constraint. The self-consistency of eq. (3) gives rise to the distributional constraint [13]

$$\left(\frac{1}{\epsilon} S_G^{(0)} + S_G^{(1)} + \epsilon S_G^{(2)} \right) \delta^{(4)}\left(\sum_a p_a\right) - \delta^{(4)}\left(\sum_a p_a + \epsilon q\right) \left(\frac{1}{\epsilon} S_G^{(0)} + S_G^{(1)} + \epsilon S_G^{(2)} \right) = \mathcal{O}(\epsilon^2). \quad (5)$$

The expansion of $\delta^{(4)}(\sum_a p_a + \epsilon q)$ in ϵ leads to relations between the operators $S_G^{(l)}$. In particular, the terms in $S_G^{(l)}$ coupling to derivatives $\partial/\partial p_a^\mu$ become constrained by lower-order soft operators $S_G^{(l' < l)}$.

Gauge invariance. Naturally, each soft operator must be gauge invariant. For the soft leg this implies that $S_a^{(l)}$ needs to be invariant under the shift $E_{\mu\nu} \rightarrow E_{\mu\nu} + q_{(\mu} \Lambda_{\nu)}$ for $q \cdot \Lambda = 0$. This can be achieved by demanding

$$q \cdot \frac{\partial}{\partial E} S_G^{(l)} \sim 0, \quad (6)$$

where the symbol \sim indicates vanishing modulo Poincaré generators

$$P^\mu = \sum_a p_a^\mu, \quad J^{\mu\nu} = \sum_a p_a^\mu \frac{\partial}{\partial p_{a\nu}} + E_a^\mu \frac{\partial}{\partial E_{a\nu}} - \mu \leftrightarrow \nu. \quad (7)$$

Notice that eq. (6) is a necessary but not sufficient condition for gauge invariance, because it corresponds to the choice $\Lambda_\mu = E_\mu$. Similarly, a gauge transformation on the soft leg a may be represented by the generator

$$W_a := p_a \cdot \frac{\partial}{\partial E_a} . \quad (8)$$

As $W_a \mathcal{M}_n = 0$ the gauge invariance of \mathcal{M}_{n+1} implies the vanishing of the commutator³

$$\left[W_a, S_G^{(l)} \right] \sim 0 , \quad (9)$$

which further constrains the form of the soft operators. Here, the zero in (9) is modulo Poincaré and gauge transformations which annihilate \mathcal{M}_n . In ref. [13] this constraint was satisfied by considering a particular combination of differential operators to appear in the soft theorems:

$$\Lambda_a^{\mu\nu} := p_a^\mu \frac{\partial}{\partial p_{a\nu}} + E_a^\mu \frac{\partial}{\partial E_{a\nu}} . \quad (10)$$

In the current article, however, we will stay more general and refrain from using this assumption.

Mass dimensions and loop counting. Finally, we need to consider the correct mass dimensions for the soft operators. In four-dimensional gravity $S_G^{(n)}$ ought to have vanishing mass dimension. This is very important to keep in mind when considering loop corrections in four dimensions, as the coupling constant κ is dimensionful. Accordingly, for every loop order an extra factor of $\kappa^2 p_a \cdot q$ appears.

2.2 Ansätze

Let us start by noting Weinberg's leading soft function

$$S_G^{(0) \text{ tree}} = \sum_{a=1}^n \frac{E_{\mu\nu} p_a^\mu p_a^\nu}{p_a \cdot q} . \quad (11)$$

We then write down the most general ansätze for the sub- and sub-subleading soft operators compatible with the above constraints. The distributional constraint eq. (5) and the form of eq. (11) require $S_G^{(1)}$ to be a differential operator of first order in p_a whereas $S_G^{(2)}$ should be of second order in p_a . Hence, the building blocks for the ansätze are single- and double-derivative operators of the schematic form

$$\begin{aligned} \text{SD}(r, s) &:= \sum_a (p_a \cdot q)^{-r} \left[(V \cdot E) (V \cdot E) (V^\mu V^\nu L_{\mu\nu}) \right]_{\mathcal{O}(q^s)} , \\ \text{DD}(r, s) &:= \sum_a (p_a \cdot q)^{-r} \left[(V \cdot E) (V \cdot E) (V^\mu V^\nu L_{\mu\nu}) (V^\rho V^\kappa L_{\rho\kappa}) \right]_{\mathcal{O}(q^s)} , \end{aligned} \quad (12)$$

³Strictly speaking, this is only a sufficient condition for $S_G^{(l)}$ to be gauge invariant. Nevertheless, we will see that the results in the following sections vanish upon the substitution $E_{a\mu} E_{a\nu} \rightarrow \Lambda_{a,(\mu} p_{a,\nu)}$ with $\Lambda_a \cdot p_a = 0$.

	Tree	1-loop	2-loop	3-loop
$S_G^{(0)}$	SD(2, 1), DD(3, 2)	DD(2, 1)	–	–
$S_G^{(1)}$	<u>SD(2, 2)</u> , <u>DD(3, 3)</u>	SD(1, 1), DD(2, 2)	DD(1, 1)	–
$S_G^{(2)}$	<u>SD(2, 3)</u> , <u>DD(3, 4)</u>	<u>SD(1, 2)</u> , <u>DD(2, 3)</u>	SD(0, 1), DD(1, 2)	DD(0, 1)

Table 1: Potential single- and double-derivative contributions to the soft operators permitted by dimensional analysis, soft scaling and loop counting. Only the underlined terms turn out to be non-vanishing – the others vanish by either constraints or explicit computation. There are no higher loop contributions.

where the tensor $L_{\mu\nu}$ can take the values

$$L_{\mu\nu} \in \left\{ p_{a\mu} \frac{\partial}{\partial p_a^\nu}, E_{a\mu} \frac{\partial}{\partial E_a^\nu} \right\}. \quad (13)$$

Moreover, the vector V takes one of the three values

$$V^\mu \in \{ p_a^\mu, n^\mu \sqrt{p_a \cdot q}, q^\mu \}, \quad (14)$$

where n is a “dummy” index vector waiting to be contracted with itself as in

$$(A \cdot n) (B \cdot n) \rightarrow A \cdot B. \quad (15)$$

Obviously, only even powers of n are kept in the ansätze of eqs. (12). Multiple occurrences of n are allowed and encode all possible contractions weighted with independent coefficients. In this manner we generate all allowed terms.

The number s in eq. (12) counts the effective power of q in the square brackets. Note that $s \in [1, 3]$ for $\text{SD}(r, s)$ and $s \in [1, 5]$ for $\text{DD}(r, s)$. The allowed terms emerging from eq. (12) are severely reduced by the on-shell conditions $E \cdot E = E \cdot q = E_a \cdot E_a = E_a \cdot p_a = 0$. Thus armed, it is straightforward to identify the potential single- and double-derivative contributions to the soft operators $S_G^{(0)}$, $S_G^{(1)}$, $S_G^{(2)}$ at tree and loop level, see table 1.

Finally let us comment on the possibility of a local soft operator without any derivative terms. It needs to have the form (cf. eq. (12))

$$\text{ND}(r, 0) = \sum_a \frac{(E \cdot p_a)^2}{(p_a \cdot q)^r}, \quad (16)$$

because the only admissible value for V is p_a . Imposing gauge invariance on the soft leg, i.e. acting with $q \cdot \partial_E$, immediately restricts the invariant operators to $r = 1$, which is the Weinberg soft factor eq. (11). Hence, we conclude that there is no other possible no-derivative structure for $S_G^{(l)}$.

2.3 Tree level results

After applying the constraints, the tree level results read

$$S_G^{(0) \text{ tree}} = \sum_{a=1}^n \frac{E_{\mu\nu} p_a^\mu p_a^\nu}{p_a \cdot q}, \quad (17)$$

$$\begin{aligned} S_G^{(1) \text{ tree}} = & \sum_{a=1}^n \frac{(p_a \cdot E) E_\rho q_\sigma}{p_a \cdot q} J_a^{\rho\sigma} \\ & + \tilde{c} \sum_{a=1}^n \left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right) \left[\frac{p_a \cdot E}{p_a \cdot q} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right], \\ & + d_1 \sum_{a=1}^n \left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right)^2 \frac{\partial}{\partial E_a} \cdot \frac{\partial}{\partial E_a} \\ & + d_2 \sum_{a=1}^n \frac{1}{p \cdot q} \left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right)^2 (p_a \cdot \frac{\partial}{\partial E_a}) (q \cdot \frac{\partial}{\partial E_a}) \end{aligned} \quad (18)$$

$$\begin{aligned} S_G^{(2) \text{ tree}} = & \frac{1}{2} \sum_{a=1}^n \frac{1}{q \cdot p_a} E^{\lambda\sigma} q^\rho q^\gamma J_{a,\rho\sigma} J_{a,\gamma\lambda} + \\ & + c_1 \sum_{a=1}^n \frac{1}{q \cdot p_a} \left(\frac{(p_a \cdot E)(q \cdot E_a)}{q \cdot p_a} - E \cdot E_a \right)^2 \left(q \cdot \frac{\partial}{\partial E_a} \right)^2 \\ & + \tilde{c} \sum_{a=1}^n \left[\frac{(p_a \cdot E)(q \cdot E_a)}{q \cdot p_a} - E \cdot E_a \right] \left[\frac{E \cdot p_a}{q \cdot p_a} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right] \left[\frac{E_a \cdot q}{q \cdot p_a} q \cdot \frac{\partial}{\partial E_a} + q \cdot \frac{\partial}{\partial p_a} \right] \end{aligned} \quad (19)$$

with the Lorentz generator density

$$J_a^{\rho\sigma} = \left(p_a^\rho \frac{\partial}{\partial p_{a\sigma}} + E_a^\rho \frac{\partial}{\partial E_{a\sigma}} - \rho \leftrightarrow \sigma \right) \quad (20)$$

and four undetermined coefficients \tilde{c}, c_1, d_1, d_2 . In ref. [13] the constants \tilde{c}, c_1 were claimed to a priori be different for each hard leg. However, since gravity amplitudes are invariant under any permutation of the external legs, the above soft factors are compatible with this additional symmetry only if \tilde{c} and c_1 agree for all hard legs. Furthermore, the terms proportional to d_1 and d_2 were not displayed in [13], because double-derivative terms for the subleading operator were not considered there. Finally, notice that this result is derived without any reference to the precise form of the amplitudes \mathcal{M}_{n+1} and \mathcal{M}_n .

The only features used are gauge and Lorentz invariance together with total momentum conservation (ensured by an overall momentum-conserving delta function). The above results apply in particular to hard amplitudes $\mathcal{M}_n(p_1, \dots, p_n)$ involving scalars or photons along with gravitons. While for a hard scalar the single derivative ∂_{E_a} vanishes, for a hard photon only the double derivative $\partial_{E_a}^2$ does.

In ref. [4], it was shown that $\tilde{c} = d_1 = d_2 = c_1 = 0$ for tree-level pure-graviton amplitudes in four dimensions.

3 Soft-graviton operators at loop level

Let us now study the admissible soft operators at loop level. Considering loops requires taking the mass dimension of the four-dimensional gravitational coupling $[\kappa] = [p^{-1}] = -1$ into account. Following the derivation of [13] outlined above, we find several additional permissible terms in the ansätze. A naïve way to produce consistent higher loop corrections satisfying gauge invariance and obeying the distributional constraints consists of promoting $S_G^{(n) \text{ } l\text{-loop}}$ to $S_G^{(n+1) \text{ } (l+1)\text{-loop}}$ by multiplication with $\kappa^2 (p_a \cdot q)$. However, it is quickly checked that the terms proportional to $J_a^{\mu\nu}$ in eqns. (18) as well as the promotion of $S_G^{(0)}$ violate gauge invariance.

3.1 Vanishing of the highest-loop contributions

As stated above, there are no loop corrections to $S_G^{(0)}$ [16]. Employing our method, this can be easily rederived for the local and universal factorizing contributions by looking at table 1: The potential tree-level derivative operators $\text{SD}(2, 1)$ and $\text{DD}(3, 2)$ vanish by explicit computation. The double-derivative candidate for a one-loop contribution to $S_G^{(0)}$ takes the form

$$\text{DD}(2, 1) = \frac{\kappa^2 (p_a \cdot E)^2}{p_a \cdot q} L_\nu^\mu L_{\mu\rho} p_a^\nu p_a^\rho. \quad (21)$$

Upon inserting either form of $L_{\mu\nu}$ in eq. (13) we see that $\text{DD}(2, 1)$ vanishes by virtue of the on-shell relations $E_a \cdot E_a = p_a \cdot p_a = E_a \cdot p_a = 0$. Hence there are no factorizing loop corrections to the leading soft-graviton behavior. This simple argument directly applies to the potential two-loop contribution to $S_G^{(1)}$ in $\text{DD}(1, 1)$ as well as to the potential three-loop contribution to $S_G^{(2)}$ in $\text{DD}(0, 1)$. Since they differ by one and two factors of $\kappa^2 (p_a \cdot q)$ from eq. (21) respectively, they vanish by the same reasoning.

3.2 Non-vanishing loop contributions

The absence of loop corrections to the leading soft operator $S_G^{(0)}$ has an immediate consequence for the possible form of the subleading soft operators at higher loop orders. The distributional constraint eq. (5) applied to the higher-loop contributions entails that the subleading soft operators $S_G^{(1)}$ and $S_G^{(2)}$ may not contain momentum derivatives terms but only derivatives with respect to the hard polarizations E_a . This further restricts our ansätze and leads to the following results: For the subleading soft operator $S_G^{(1)}$ *all* possible loop contributions vanish because of incompatibility with the constraints.

Turning to the sub-subleading soft operator $S_G^{(2)}$ we find that there are non-vanishing local and universal factorizing contributions at one-loop order exclusively. We find *four* contributions to $S_G^{(2) \text{ } 1\text{-loop}}$, all of which originate from the tree-level contributions to $S_G^{(1) \text{ tree}}$ in eq. (18) by

multiplying with $\kappa^2 (p_a \cdot q)$ as explained above. Collecting everything, we find

$$\begin{aligned}
S_G^{(2) \text{ 1-loop}} = & \kappa^2 f_1 \sum_{a=1}^n (p_a \cdot q) \left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right) \left[\frac{p_a \cdot E}{p_a \cdot q} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right] \\
& + \kappa^2 f_2 \sum_{a=1}^n (p_a \cdot q) \left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right)^2 \frac{\partial}{\partial E_a} \cdot \frac{\partial}{\partial E_a} \\
& + \kappa^2 f_3 \sum_{a=1}^n \left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right)^2 (p_a \cdot \frac{\partial}{\partial E_a}) (q \cdot \frac{\partial}{\partial E_a}) \\
& + \kappa^2 f_4 \sum_{a=1}^n \left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right) \left[E_a \cdot q p_a \cdot \frac{\partial}{\partial E_a} - p_a \cdot q E_a \cdot \frac{\partial}{\partial E_a} \right] \times \\
& \times \left[\frac{p_a \cdot E}{p_a \cdot q} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right]. \tag{22}
\end{aligned}$$

As a matter of fact, the term proportional to f_4 is effectively equal to the term proportional to f_1 because $E_a \partial_{E_a} \mathcal{M} = 2\mathcal{M}$ and the remaining $p_a \cdot \partial_{E_a}$ acts – together with the further differential operator – as a gauge transformation. Hence we may set $f_4 = 0$ ⁴.

Beyond these three terms *all* local and universal factorizing loop corrections to the soft operators vanish. In particular there cannot be any factorizing loop corrections to $S_G^{(0)}$ or $S_G^{(1)}$. Moreover, the factorizing loop corrections to $S_G^{(2)}$ are one-loop exact as shown above.

4 Rewriting polarization derivatives

All loop corrections appear in the form of polarization derivatives. Using the completeness relation [21]

$$\eta^{\mu\nu} = -E_a^\mu \bar{E}_a^\nu - \bar{E}_a^\mu E_a^\nu + \frac{p_a^\mu r^\nu + r^\mu p_a^\nu}{r \cdot p_a}, \tag{23}$$

these can be simplified and expressed in a more convenient form. Here \bar{E}_a^μ is the polarization of opposite helicity compared to E_a^μ and r^μ is the reference vector needed to define the polarizations⁵. Using this language, the n -graviton amplitude can be rewritten as

$$\mathcal{M}(p_a, p_1, \dots, p_{n-1}) = E_a^\mu E_a^\nu \mathcal{M}_{\mu\nu}(p_a, p_1, \dots, p_{n-1}). \tag{24}$$

Accordingly, we define the conjugate-helicity amplitude as well as the effective scalar-graviton amplitude as

$$\begin{aligned}
\bar{\mathcal{M}} &:= \bar{E}_a^\mu \bar{E}_a^\nu \mathcal{M}_{\mu\nu}(p_a, p_1, \dots, p_{n-1}), \\
\mathcal{M}_o &:= \bar{E}_a^\mu E_a^\nu \mathcal{M}_{\mu\nu}(p_a, p_1, \dots, p_{n-1}) = -\frac{1}{2} \mathcal{M}_\mu^\mu + \frac{1}{r \cdot p_a} p_a^\mu r^\nu \mathcal{M}_{\mu\nu}. \tag{25}
\end{aligned}$$

⁴Notice that the term proportional to f_4 already originates at tree-level, but can be reabsorbed in other terms in a way similar to what explained before.

⁵Notice that $E_a \cdot \bar{E}_a = -1$.

The last relation follows from inserting the completeness relation eq. (23) by virtue of $\mathcal{M}_{\mu\nu} = \mathcal{M}_{\nu\mu}$. Using the above notation, we find after dropping the arguments of the amplitudes

$$\partial_{E_a^\mu} \mathcal{M} = -2\bar{E}_a^\mu \mathcal{M} - 2E_a^\mu \mathcal{M}_\circ + 2\frac{p_a^\mu}{p_a \cdot r} (E_a^\rho r^\kappa \mathcal{M}_{\rho\kappa}), \quad (26)$$

$$\begin{aligned} \partial_{E_a^\mu} \partial_{E_a^\nu} \mathcal{M} &= 2E_a^\mu E_a^\nu \bar{\mathcal{M}} + 2\bar{E}_a^\mu \bar{E}_a^\nu \mathcal{M} + 4E_a^{(\mu} \bar{E}_a^{\nu)} \mathcal{M}_\circ - 4\frac{E_a^{(\mu} p_a^{\nu)}}{r \cdot p_a} (\bar{E}_a^\rho r^\kappa \mathcal{M}_{\rho\kappa}) \\ &\quad - 4\frac{\bar{E}_a^{(\mu} p_a^{\nu)}}{r \cdot p_a} (E_a^\rho r^\kappa \mathcal{M}_{\rho\kappa}) + 2\frac{p_a^{(\mu} p_a^{\nu)}}{(r \cdot p_a)^2} (r^\rho r^\kappa \mathcal{M}_{\rho\kappa}) + 4\frac{p_a^{(\mu} r^{\nu)}}{(r \cdot p_a)^2} (p_a^\rho r^\kappa \mathcal{M}_{\rho\kappa}). \end{aligned} \quad (27)$$

The particular combinations appearing in the soft operators can now be expressed in the following way

$$\left[\frac{p_a \cdot E}{p_a \cdot q} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right] \mathcal{M} = -2\bar{T}_a \mathcal{M} - 2T_a \mathcal{M}_\circ, \quad (28)$$

$$\frac{\partial}{\partial E_a} \cdot \frac{\partial}{\partial E_a} \mathcal{M} = 2\mathcal{M}_\mu^\mu, \quad (29)$$

where T_a and \bar{T}_a are the gauge invariant quantities

$$T_a := \frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a, \quad \bar{T}_a := \frac{(E \cdot p_a)(\bar{E}_a \cdot q)}{p_a \cdot q} - E \cdot \bar{E}_a. \quad (30)$$

An important relation is

$$T_a \bar{T}_a = 0, \quad (31)$$

which is most easily seen by choosing q as the reference vector for E_a and \bar{E}_a . Then $T_a \bar{T}_a = (E \cdot E_a)(E \cdot \bar{E}_a)$. The later expression always vanishes: depending on the soft and hard helicities either the first or the second factor is zero⁶. Using these relations one may now simplify the soft-operators at tree and loop level obtained in the previous sections.

Tree level. For the novel tree-level terms in our soft operators we find at subleading order

$$\Delta S_G^{(1) \text{ tree}} \mathcal{M} = \tilde{c} \sum_{a=1}^n T_a \left[\frac{p_a \cdot E}{p_a \cdot q} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right] \mathcal{M} = -2\tilde{c} \sum_{a=1}^n T_a^2 \mathcal{M}_\circ. \quad (33)$$

⁶ In spinor helicity language one finds

$$T_a = \begin{cases} h_q = +2, h_a = +2 : & = -2 \frac{[qa]}{\langle qa \rangle} \\ h_q = -2, h_a = -2 : & = -2 \frac{\langle qa \rangle}{[qa]} \\ \text{else :} & = 0 \end{cases}. \quad (32)$$

The contributions proportional to $d_{1,2}$ are of the same form as the loop-level contribution to $S_G^{(2) \text{ 1-loop}}$ which will be discussed below. The remaining undetermined terms of the sub-subleading soft operator parametrized by c_1 and \tilde{c} read

$$\Delta_1 S_G^{(2) \text{ tree}} \mathcal{M} = c_1 \sum_{a=1}^n \frac{1}{q \cdot p_a} T_a^2 \left(q \cdot \frac{\partial}{\partial E_a} \right)^2 \mathcal{M} = 2 c_1 \sum_{a=1}^n \frac{1}{q \cdot p_a} T_a^2 (q^\mu q^\nu \mathcal{M}_{\mu\nu}) \quad (34)$$

$$\begin{aligned} \Delta_2 S_G^{(2) \text{ tree}} \mathcal{M} &= \tilde{c} \sum_{a=1}^n T_a \left(\frac{E_a \cdot q}{q \cdot p_a} q \cdot \frac{\partial}{\partial E_a} + q \cdot \frac{\partial}{\partial p_a} \right) \left(\frac{E \cdot p_a}{q \cdot p_a} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right) \mathcal{M} \\ &= \tilde{c} \sum_{a=1}^n T_a \left(\frac{E_a \cdot q}{q \cdot p_a} q \cdot \frac{\partial}{\partial E_a} + q \cdot \frac{\partial}{\partial p_a} \right) \left(-2\bar{T}_a \mathcal{M} - 2T_a \mathcal{M}_o \right) \\ &= -2\tilde{c} \sum_{a=1}^n T_a^2 \left(\frac{E_a \cdot q}{q \cdot p_a} q \cdot \frac{\partial}{\partial E_a} + q \cdot \frac{\partial}{\partial p_a} \right) \mathcal{M}_o, \end{aligned} \quad (35)$$

where in the last step we used the identities $q \cdot \partial_{E_a} T_a = 0 = q \cdot \partial_{E_a} \bar{T}_a$ and $q \cdot \partial_{p_a} T_a = 0 = q \cdot \partial_{p_a} \bar{T}_a$ as well as $T_a \bar{T}_a = 0$.

Loop level. The loop corrections in eq. (22) can be rewritten as

$$\begin{aligned} S_G^{(2) \text{ 1-loop}} \mathcal{M} &= \kappa^2 f_1 \sum_{a=1}^n (p_a \cdot q) T_a \left(-2\bar{T}_a \mathcal{M} - 2T_a \mathcal{M}_o \right) + \kappa^2 f_2 \sum_{a=1}^n (p_a \cdot q) T_a^2 \left(2\mathcal{M}_\mu^\mu \right) \\ &\quad + 2\kappa^2 f_3 \sum_{a=1}^n T_a^2 (p_a^\mu q^\nu \mathcal{M}_{\mu\nu}) \\ &= -2\kappa^2 f_1 \sum_{a=1}^n T_a^2 (p_a \cdot q) \mathcal{M}_o + 2\kappa^2 f_2 \sum_{a=1}^n (p_a \cdot q) T_a^2 \mathcal{M}_\mu^\mu \\ &\quad + 2\kappa^2 f_3 \sum_{a=1}^n T_a^2 (p_a^\mu q^\nu \mathcal{M}_{\mu\nu}), \end{aligned} \quad (36)$$

which can be further simplified by choosing the reference momentum to equal q . This is possible because T_a is gauge invariant by itself. After all, we arrive at the final result

$$S_G^{(2) \text{ 1-loop}} \mathcal{M} = \kappa^2 e_1 \sum_{a=1}^n T_a^2 (p_a^\mu q^\nu \mathcal{M}_{\mu\nu}) + \kappa^2 e_2 \sum_{a=1}^n (p_a \cdot q) T_a^2 \mathcal{M}_\mu^\mu. \quad (37)$$

Interpretation. The loop correction couples to $p_a^\mu q^\nu \mathcal{M}_{\mu\nu}$ and \mathcal{M}_μ^μ exclusively. Note that the latter correction is precisely the structure that was uncovered in an explicit computation for a ϕR^2 higher-derivative gravity theory where \mathcal{M}_o is a single-scalar $(n-1)$ -graviton amplitude [9]. To our knowledge there is no general theorem stating that the polarization-stripped and traced amplitude \mathcal{M}_μ^μ vanishes in pure gravity at loop level. While it does so for four gravitons at tree level, which can be tested using the explicit results in ref. [22], calculations are involved already at the one-loop level. At least the trace of the four-point integrand does not vanish.

5 Discussion

In this paper we have determined all admissible soft factors for local and universal factorizing loop corrections in four-dimensional gravity.

We found that only the sub-subleading soft operator receives loop corrections. It turns out that these corrections are one-loop exact and have two contributions proportional to $p_a^\mu q^\nu \mathcal{M}_{\mu\nu}$ and \mathcal{M}_μ^μ . This is consistent with the results in ref. [11], where loop corrections arising from virtual scalars in graviton amplitudes were shown to not contribute to subleading soft factors.

Hence the local and universal factorizing contributions to the soft graviton operators do not receive corrections at first and second perturbative order. This implies that the soft-graviton Ward identities of extended BMS symmetry do not suffer from anomalies caused by local universal operators in the factorized sector studied here. They will be affected, however, by the non-factorized contributions entangled with the infrared divergences. It would be interesting to understand this sector in detail as well.

While the tensorial form of the loop contributions to the sub-subleading soft graviton theorems is fixed by our analysis, the undetermined scalar prefactors will reflect the field content of the theory. For example, our coupling to \mathcal{M}_μ^μ exactly matches the structure found in [9] for ϕR^2 gravity.

Finally, let us once more comment on gauge theory in four dimensions where the coupling constant is dimensionless. Therefore, the loop corrections simply correspond to the undetermined part of the subleading soft operator $S_{\text{YM}}^{(1)}$ of section 3 in ref. [13], which in turn agrees with the one-loop correction to $S_{\text{YM}}^{(1)}$ quoted in [11]. The undetermined coefficient again depends on the matter content of the gauge theory in question. The tensorial form of the soft operators is, however, universal.

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References

- [1] A. Strominger, “*On BMS Invariance of Gravitational Scattering*”, [arxiv:1312.2229](#). • T. He, V. Lysov, P. Mitra and A. Strominger, “*BMS supertranslations and Weinberg’s soft graviton theorem*”, [arxiv:1401.7026](#). • D. Kapec, V. Lysov, S. Pasterski and A. Strominger, “*Semiclassical Virasoro Symmetry of the Quantum Gravity S-Matrix*”, [arxiv:1406.3312](#).
- [2] H. Bondi, M. van der Burg and A. Metzner, “*Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*”, *Proc.Roy.Soc.Lond.* A269, 21 (1962). • R. Sachs,

- “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times”, Proc.Roy.Soc.Lond. A270, 103 (1962). • G. Barnich and C. Troessaert, “Aspects of the BMS/CFT correspondence”, JHEP 1005, 062 (2010), [arxiv:1001.1541](#).
- [3] S. Weinberg, “Photons and Gravitons in s Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass”, Phys.Rev. 135, B1049 (1964).
 - [4] F. Cachazo and A. Strominger, “Evidence for a New Soft Graviton Theorem”, [arxiv:1404.4091](#).
 - [5] B. U. W. Schwab and A. Volovich, “Subleading soft theorem in arbitrary dimension from scattering equations”, [arxiv:1404.7749](#). • N. Afkhami-Jeddi, “Soft Graviton Theorem in Arbitrary Dimensions”, [arxiv:1405.3533](#). • C. Kalousios and F. Rojas, “Next to subleading soft-graviton theorem in arbitrary dimensions”, [arxiv:1407.5982](#). • M. Zlotnikov, “Sub-sub-leading soft-graviton theorem in arbitrary dimension”, JHEP 1410, 148 (2014), [arxiv:1407.5936](#).
 - [6] F. Low, “Bremsstrahlung of very low-energy quanta in elementary particle collisions”, Phys.Rev. 110, 974 (1958). • T. Burnett and N. M. Kroll, “Extension of the low soft photon theorem”, Phys.Rev.Lett. 20, 86 (1968).
 - [7] E. Casali, “Soft sub-leading divergences in Yang-Mills amplitudes”, [arxiv:1404.5551](#).
 - [8] B. U. Schwab, “Subleading Soft Factor for String Disk Amplitudes”, JHEP 1408, 062 (2014), [arxiv:1406.4172](#).
 - [9] M. Bianchi, S. He, Y.-t. Huang and C. Wen, “More on Soft Theorems: Trees, Loops and Strings”, [arxiv:1406.5155](#).
 - [10] Y. Geyer, A. E. Lipstein and L. Mason, “Ambitwistor strings at null infinity and subleading soft limits”, [arxiv:1406.1462](#).
 - [11] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, “Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance”, Phys.Rev. D90, 084035 (2014), [arxiv:1406.6987](#).
 - [12] C. White, “Diagrammatic insights into next-to-soft corrections”, [arxiv:1406.7184](#).
 - [13] J. Broedel, M. de Leeuw, J. Plefka and M. Rosso, “Constraining subleading soft gluon and graviton theorems”, Phys.Rev. D90, 065024 (2014), [arxiv:1406.6574](#).
 - [14] Z. Bern, S. Davies and J. Nohle, “On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons”, Phys.Rev. D90, 085015 (2014), [arxiv:1405.1015](#).
 - [15] S. He, Y.-t. Huang and C. Wen, “Loop Corrections to Soft Theorems in Gauge Theories and Gravity”, [arxiv:1405.1410](#).
 - [16] S. Weinberg, “Infrared photons and gravitons”, Phys.Rev. 140, B516 (1965). • Z. Bern, L. J. Dixon, M. Perelstein and J. Rozowsky, “Multileg one loop gravity amplitudes from gauge theory”, Nucl.Phys. B546, 423 (1999), [hep-th/9811140](#).
 - [17] F. Cachazo and E. Y. Yuan, “Are Soft Theorems Renormalized?”, [arxiv:1405.3413](#).
 - [18] D. Bonocore, E. Laenen, L. Magnea, L. Vernazza and C. D. White, “The method of regions and next-to-soft corrections in Drell-Yan production”, [arxiv:1410.6406](#).
 - [19] Z. Bern and G. Chalmers, “Factorization in one loop gauge theory”, Nucl.Phys. B447, 465 (1995), [hep-ph/9503236](#).

- [20] D. A. Kosower and P. Uwer, “*One loop splitting amplitudes in gauge theory*”, Nucl.Phys. B563, 477 (1999), hep-ph/9903515. • D. A. Kosower, “*All orders singular emission in gauge theories*”, Phys.Rev.Lett. 91, 061602 (2003), hep-ph/0301069.
- [21] L. J. Dixon, “*Calculating scattering amplitudes efficiently*”, hep-ph/9601359.
- [22] S. Sannan, “*Gravity as the Limit of the Type II Superstring Theory*”, Phys.Rev. D34, 1749 (1986).